

THE SELF-LINKING NUMBER OF A CLOSED CURVE IN \mathbb{R}^n

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ABSTRACT. We introduce the self-linking number of a smooth closed curve $\alpha : S^1 \rightarrow \mathbb{R}^n$ with respect to a 3-dimensional vector bundle over the curve, provided that some regularity conditions are satisfied. When $n = 3$, this construction gives the classical self-linking number of a closed embedded curve with non-vanishing curvature [5]. We also look at some interesting particular cases, which correspond to the osculating or the orthogonal vector bundle of the curve.

1. INTRODUCTION

It is well known that two closed embedded curves $\alpha, \beta : S^1 \rightarrow \mathbb{R}^3$ are equivalent as knots if and only if there is a continuous map $H : S^1 \times [0, 1] \rightarrow \mathbb{R}^3$ such that for any $u \in [0, 1]$, the curve $H_u : S^1 \rightarrow \mathbb{R}^3$ given by $H_u(t) = H(t, u)$ is an embedding and $H_0 = \alpha, H_1 = \beta$ (such a map H is said to be an *isotopy* between α, β). For instance, if we look at the two curves shown in Figure 1, it follows that they are equivalent as knots (in fact, they are equivalent to the trivial knot).

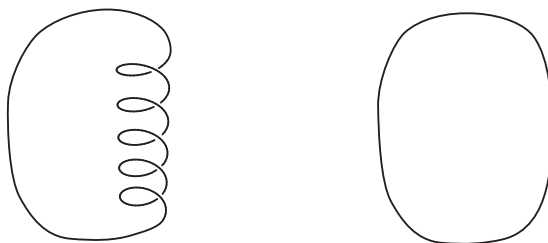


FIGURE 1.

However, suppose that we construct these two curves so that they are of class C^3 and have non-vanishing curvature at each point. Then, it is not difficult to see that it is not possible to have an isotopy H such that for any $u \in [0, 1]$, H_u has the same property (such a map will be called a *non-degenerate isotopy*). This is due to the fact that these two curves have different self-linking number. This number was introduced by Călugăreanu [1] and studied with more detail by Pohl [5]. It can be seen as the linking number between the given curve and a curve obtained by slightly pushing the curve along the principal normal. Moreover, it is possible to compute the self-linking number by means of the following integral formula:

$$SL(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} \frac{\det(\alpha(s) - \alpha(t), \alpha'(s), \alpha'(t))}{\|\alpha(s) - \alpha(t)\|^3} dt \wedge ds + \frac{1}{2\pi} \int_{S^1} \tau dt,$$

where τ is the torsion of α . Recently, Gluck and Pan [2] have shown that there is a non-degenerate isotopy between two embedded closed curves with non-vanishing curvature in \mathbb{R}^3 if and only if they have the same knot type and the same self-linking number. Thus,

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the self-linking number is the key invariant if we want to do a curvature sensitive version of the knot theory.

In this paper, we propose a generalization of this invariant for the case of a closed smooth curve $\alpha : S^1 \rightarrow \mathbb{R}^n$. In our construction, we have to choose a 3-dimensional vector bundle over the curve, so that some regularity conditions hold between the curve and the vector bundle. In the last part of the paper, we analyze the osculating and the orthogonal self-linking number, which correspond to the cases where the vector bundle is the osculating or the orthogonal vector bundle of the curve, respectively. These numbers can be also interpreted in terms of intersection numbers of the curve with the orthogonal or the osculating developable hypersurface of the curve, respectively. In particular, it seems possible to relate them to some bitangency properties of the curve [4].

A different approach in generalizing the self-linking number can be found in [6], where it is considered a smooth map $f : M \rightarrow \mathbb{R}^{2n+1}$ from a closed orientable smooth n -manifold M into \mathbb{R}^{2n+1} .

2. THE LINKING NUMBER OF TWO CURVES WITH RESPECT TO A VECTOR BUNDLE

The linking number of two disjoint closed curves $\alpha, \beta : S^1 \rightarrow \mathbb{R}^3$ is a well known invariant, which is defined as the degree of the map $e_1 : S^1 \times S^1 \rightarrow S^2$ given by $e_1(t, s) = (\beta(s) - \alpha(t)) / \|\beta(s) - \alpha(t)\|$. In this section, we will generalize this concept for two closed curves in \mathbb{R}^n , by using a vector bundle over one of them.

Let $\nu : E \rightarrow S^1$ be a smooth 3-dimensional oriented vector subbundle of the trivial vector bundle $S^1 \times \mathbb{R}^n \rightarrow S^1$. Given $t \in S^1$, we will denote the fiber by ν_t , which is a 3-dimensional vector subspace of \mathbb{R}^n . Moreover, we will put $n_t : \mathbb{R}^n \rightarrow \nu_t$ and $o_t : \mathbb{R}^n \rightarrow \nu_t^\perp$ for the orthogonal projections, where ν_t^\perp is the orthogonal subspace to ν_t . Using these projections, we define the *covariant derivative* of a section $h : S^1 \rightarrow \mathbb{R}^n$ of ν by

$$(Dh)(t) = n_t(h'(t)) = h'(t) - o_t(h'(t)),$$

and we will say that h is *parallel* if $Dh = 0$.

Now, if we fix a parameterization of S^1 in the interval $[0, \ell]$, we can solve the equations of parallel transport and consider $\{p_1(t), p_2(t), p_3(t)\}$, a parallel oriented orthonormal frame of ν_t , for $t \in [0, \ell]$ (so that in general $p_i(0)$ can be distinct from $p_i(\ell)$). This frame allows us to define the linear map $c_t : \mathbb{R}^n \rightarrow \mathbb{R}^3$ by

$$c_t(x) = (\langle x, p_1(t) \rangle, \langle x, p_2(t) \rangle, \langle x, p_3(t) \rangle),$$

so that the restriction of c_t to ν_t is an oriented isometry.

Given $h : S^1 \rightarrow \mathbb{R}^n$ any smooth map, it will be useful to know about the derivative of $c_t h(t)$. Let $\{u_i\}_{i=1}^3$ denote the canonical basis of \mathbb{R}^3 . Then,

$$\begin{aligned} (c_t h(t))' &= \sum_{i=1}^3 \langle h(t), p_i(t) \rangle' u_i = \sum_{i=1}^3 (\langle h'(t), p_i(t) \rangle + \langle h(t), p_i'(t) \rangle) u_i \\ &= c_t h'(t) + \sum_{i=1}^3 \langle h(t), o_t p_i'(t) \rangle c_t p_i(t) = c_t(h'(t) + A_t h(t)), \end{aligned}$$

where $A_t : \mathbb{R}^n \rightarrow \nu_t$ is the linear map given by

$$A_t(x) = \sum_{i=1}^3 \langle x, o_t p_i'(t) \rangle p_i(t).$$

It is not difficult to see that A_t does not depend on the chosen orthonormal frame $p_i(t)$, parallel or not.

Definition 2.1. Let $\alpha, \beta : S^1 \rightarrow \mathbb{R}^n$ be two smooth closed curves in \mathbb{R}^n and suppose that $\beta(s) - \alpha(t) \notin \nu_t^\perp$ for any $(t, s) \in S^1 \times S^1$. We define the map $e_1 : [0, \ell] \times S^1 \rightarrow S^2$ by

$$e_1(t, s) = \frac{c_t(\beta(s) - \alpha(t))}{\|c_t(\beta(s) - \alpha(t))\|}.$$

Note that the imposed condition on the curves implies that $c_t(\beta(s) - \alpha(t)) \neq 0$ and thus, e_1 is well defined.

Lemma 2.2. Let Ω_2 be the standard volume form on S^2 . Then $e_1^* \Omega_2$ does not depend on the frame $p_i(t)$ and it defines a closed smooth 2-form on $S^1 \times S^1$.

Proof. To abbreviate, we denote $\delta(t, s) = \beta(s) - \alpha(t)$. Then,

$$e_1^* \Omega_2 = \frac{\det(c_t \delta(t, s), \partial_t c_t \delta(t, s), \partial_s c_t \delta(t, s))}{\|c_t \delta(t, s)\|^3} dt \wedge ds,$$

where ∂_t and ∂_s denote the partial derivatives with respect to t and s respectively. But, according to the above computation, $\partial_t c_t \delta(t, s) = c_t(-\alpha'(t) + A_t \delta(t, s))$ and $\partial_s c_t \delta(t, s) = c_t \beta'(s)$. Therefore,

$$\begin{aligned} e_1^* \Omega_2 &= \frac{\det(c_t \delta(t, s), c_t(-\alpha'(t) + A_t \delta(t, s)), c_t \beta'(s))}{\|c_t \delta(t, s)\|^3} dt \wedge ds \\ &= \frac{\det(n_t \delta(t, s), n_t(-\alpha'(t) + A_t \delta(t, s)), n_t \beta'(s))}{\|n_t \delta(t, s)\|^3} dt \wedge ds, \end{aligned}$$

where the last determinant has to be considered with respect to any oriented orthonormal frame of ν_t . \square

Definition 2.3. Let $\alpha, \beta : S^1 \rightarrow \mathbb{R}^n$ be two smooth closed curves in \mathbb{R}^n and suppose that $\beta(s) - \alpha(t) \notin \nu_t^\perp$ for any $(t, s) \in S^1 \times S^1$. We define its *linking number* with respect to ν as

$$L_\nu(\alpha, \beta) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2.$$

It follows from this definition that if ν_t is constant, then $L_\nu(\alpha, \beta)$ coincides with the classical linking number of the projected curves, $L(c_t \circ \alpha, c_t \circ \beta)$. In particular, when $n = 3$, we have that $L_\nu(\alpha, \beta) = L(\alpha, \beta)$, because then ν is the trivial bundle.

Lemma 2.4. Let α, β, ν be as in Definition 2.3. Then, there is $\tilde{\beta} : D^2 \rightarrow \mathbb{R}^n$ extension of β , such that $\tilde{\beta}(z) - \alpha(t) \notin \nu_t^\perp$ except for a finite number of pairs $(t, z) \in S^1 \times D^2$.

Proof. Let $\beta_1 : D^2 \rightarrow \mathbb{R}^n$ be an arbitrary extension of β . Then the map $F : S^1 \times D^2 \times \mathbb{R}^n \rightarrow S^1 \times D^2 \times \mathbb{R}^n$ given by $F(t, z, x) = (t, z, \beta_1(z) - \alpha(t) + x)$ is a diffeomorphism. In particular, it is transverse to the submanifold $W = \{(t, z, x) : x \in \nu_t^\perp\}$. By the Transversality Theorem, it follows that for almost any $x \in \mathbb{R}^n$, the map $F_x : S^1 \times D^2 \rightarrow S^1 \times D^2 \times \mathbb{R}^n$ given by $F_x(t, z) = F(t, z, x)$ is also transverse to W . Since W has codimension 3, this implies that $F_x^{-1}(W)$ is finite.

To construct the required extension $\tilde{\beta}$, we piece together β_1 near S^1 and $\beta_1 + x$ on the interior as follows. Let ϵ, δ be such that $0 < \delta < \epsilon < 1$. Let $g_{\epsilon, \delta} : D^2 \rightarrow [0, 1]$ be a smooth function such that $g_{\epsilon, \delta}(z) = 1$ if $\|z\| \leq \delta$ and $g_{\epsilon, \delta}(z) = 0$ if $\epsilon \leq \|z\| \leq 1$ and let

$\beta_{\epsilon,\delta,x}(z) = \beta_1(z) + g_{\epsilon,\delta}(z)x$. We claim that there are ϵ, δ as above and $R > 0$ such that for any $x \in \mathbb{R}^n$ with $\|x\| < R$,

$$\beta_{\epsilon,\delta,x}(z) - \alpha(t) \in \nu_t^\perp \implies \beta_1(z) + x - \alpha(t) \in \nu_t^\perp.$$

Suppose that the claim is not true. Then, if for each $n > 2$ we consider $\epsilon = 1 - 1/n$, $\delta = 1 - 2/n$ and $R = 1/n$, there are $t_n \in S^1$, $z_n \in D^2$ and $x_n \in \mathbb{R}^n$ with $\|x_n\| < 1/n$ such that

$$\beta_1(z_n) + g_{\epsilon,\delta}(z_n)x_n - \alpha(t_n) \in \nu_{t_n}^\perp, \text{ but } \beta_1(z_n) + x_n - \alpha(t_n) \notin \nu_{t_n}^\perp.$$

Thus $g_{\epsilon,\delta}(z_n) \neq 1$ so that $\|z_n\| \geq 1 - 2/n$. By taking subsequences if necessary, we can suppose that $t_n \rightarrow t_0 \in S^1$ and $z_n \rightarrow s_0 \in S^1$. Thus, we arrive to $\beta(s_0) - \alpha(t_0) \in \nu_{t_0}^\perp$, in contradiction with the hypothesis. Now, we can choose $\tilde{\beta} = \beta_{\epsilon,\delta,x}$, where x is any one of the points with $\|x\| < R$ for which $F_x^{-1}(W)$ is finite. \square

Proposition 2.5. *Let α, β, ν be as in Definition 2.3. Then, $L_\nu(\alpha, \beta) \in \mathbb{Z}$.*

Proof. Let $\tilde{\beta}$ be an extension of β such that $\tilde{\beta}(z) - \alpha(t) \notin \nu_t^\perp$ for any $(t, z) \in S^1 \times D^2 \setminus P$, being $P = \{(t_1, z_1), \dots, (t_N, z_N)\}$. Then we can extend e_1 to $\tilde{e}_1 : [0, \ell] \times D^2 \setminus P \rightarrow S^2$ by putting

$$\tilde{e}_1(t, z) = \frac{c_t(\tilde{\beta}(z) - \alpha(t))}{\|c_t(\tilde{\beta}(z) - \alpha(t))\|}.$$

As in Lemma 2.2, it follows that $\tilde{e}_1^* \Omega_2$ defines a smooth 2-form on $S^1 \times D^2 \setminus P$. Moreover, since Ω_2 is closed on S^2 , $d\tilde{e}_1^* \Omega_2 = 0$ and by Stokes Theorem,

$$0 = \int_{S^1 \times S^1} e_1^* \Omega_2 + \sum_{i=1}^N \int_{\partial B_i} \tilde{e}_1^* \Omega_2,$$

where B_i denotes a small ball centered at (t_i, z_i) in the interior of $S^1 \times D^2$ and such that $B_i \cap B_j = \emptyset$ if $i \neq j$. In particular,

$$L_\nu(\alpha, \beta) = -\frac{1}{4\pi} \sum_{i=1}^N \int_{\partial B_i} \tilde{e}_1^* \Omega_2 = -\sum_{i=1}^N \deg(\tilde{e}_1|_{\partial B_i}) \in \mathbb{Z},$$

being $\deg(\tilde{e}_1|_{\partial B_i})$ the degree of the map $\tilde{e}_1|_{\partial B_i}$. \square

An immediate consequence of this, together with the fact that $L_\nu(\alpha, \beta)$ depends continuously on α, β, ν (when we consider the corresponding C^∞ Whitney topologies), is that $L_\nu(\alpha, \beta)$ is invariant under homotopies of the curves and the vector bundle.

Corollary 2.6. *Let $\alpha_u, \beta_u : S^1 \rightarrow \mathbb{R}^n$ be 1-parameter families of curves and let $\nu_u : E_u \rightarrow S^1$ be a 1-parameter family of vector bundles, all of them depending smoothly on the parameter $u \in [0, 1]$ and such that α_u, β_u, ν_u satisfy the condition of Definition 2.3, for any $u \in [0, 1]$. Then, $L_{\nu_u}(\alpha_u, \beta_u)$ is constant on u .*

In the last part of this section, we give a characterization of the linking number that will be used in the next section. Let α, β, ν be as in Definition 2.3 and suppose that there is a vector field $\mu : S^1 \rightarrow \mathbb{R}^n$ such that $\mu(t) \notin \nu_t^\perp$, for any $t \in S^1$. Let $\{f_i(t)\}_{i=4}^n$ be an orthonormal oriented frame of ν_t^\perp , that is, the basis $(p_1(t), p_2(t), p_3(t), f_4(t), \dots, f_n(t))$ has the same orientation as the canonical basis of \mathbb{R}^n . We can define the map $\chi : S^1 \times \mathbb{R} \times \mathbb{R}^{n-3} \rightarrow \mathbb{R}^n$ by

$$\chi(t, \lambda, x_4, \dots, x_n) = \alpha(t) + \lambda\mu(t) + \sum_{i=4}^n x_i f_i(t).$$

Proposition 2.7. *Suppose that β meets the map χ transversely at a finite number of points and let*

$$P_i = \beta(s_i) \in \alpha(t_i) + \lambda_i \mu(t_i) + \nu_{t_i}^\perp, \quad i = 1, \dots, N.$$

be those points. Then,

$$L_\nu(\alpha, \beta) = \frac{1}{2} \sum_{i=1}^N \operatorname{sgn}(\lambda_i) i(\beta, \chi; P_i),$$

where $i(\beta, \chi; P_i)$ denotes the intersection number of β and χ at P_i and $\operatorname{sgn}(\lambda_i)$ is the sign of λ_i .

Proof. Let $S^0 = [0, \ell] \times S^1 \setminus \{(t_1, s_1), \dots, (t_N, s_N)\}$. For any $(t, s) \in S^0$, we have that $c_t \delta(t, s) \times c_t \mu(t) \neq 0$, where $\delta(t, s) = \beta(s) - \alpha(t)$. Thus, we can define

$$e_3(t, s) = \frac{c_t \delta(t, s) \times c_t \mu(t)}{\|c_t \delta(t, s) \times c_t \mu(t)\|}$$

and $e_2(t, s) = e_3(t, s) \times e_1(t, s)$, so that $\{e_i(t, s)\}_{i=1}^3$ is a right-handed orthonormal frame of \mathbb{R}^3 .

Now, we can consider the 1-forms on S^0 defined by $\omega_{ij} = \langle de_i, e_j \rangle$, for any $i, j = 1, 2, 3$. Since $\langle e_i, e_j \rangle = \delta_{ij}$, by taking differentials we see that $\omega_{ij} = -\omega_{ji}$. Moreover, we have that

$$\begin{aligned} e_1^* \Omega_2 &= \det(e_1, \partial_t e_1, \partial_s e_1) dt \wedge ds \\ &= (\omega_{12}(\partial_t) \omega_{13}(\partial_s) - \omega_{12}(\partial_s) \omega_{13}(\partial_t)) dt \wedge ds \\ &= \omega_{12} \wedge \omega_{13}. \end{aligned}$$

But from the fact that $dde_i = 0$, we deduce that

$$0 = \langle dde_i, e_j \rangle = d\omega_{ij} - \sum_{k=1}^3 \omega_{ik} \wedge \omega_{kj}.$$

In particular, $d\omega_{32} = \omega_{12} \wedge \omega_{13} = e_1^* \Omega_2$ and it is not difficult to see that ω_{32} defines a 1-form on $S^1 \times S^1 \setminus \{(t_1, s_1), \dots, (t_N, s_N)\}$. This gives, by Stokes Theorem, that

$$L_\nu(\alpha, \beta) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 = \frac{1}{4\pi} \lim_{\epsilon \rightarrow 0} \sum_{i=1}^N \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32},$$

where $D_\epsilon(t_i, s_i)$ denotes the disk centered at (t_i, s_i) of radius $\epsilon > 0$ in $S^1 \times S^1$. To conclude the proof, we just have to show that for any $i = 1, \dots, N$,

$$\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32} = \operatorname{sgn}(\lambda_i) i(\beta, \chi; P_i).$$

On one hand, if we put $m(t, s) = c_t \delta(t, s) \times c_t \mu(t)$, it is easy to see that the left hand side is equal to ± 1 , in accordance with the sign of $D = \det(\partial_t m(t_i, s_i), \partial_s m(t_i, s_i), c_{t_i} \delta(t_i, s_i))$. If we compute this, we get

$$\begin{aligned} c_{t_i} \delta(t_i, s_i) &= \lambda_i c_{t_i} \mu(t_i) \\ \partial_s m(t_i, s_i) &= c_{t_i} \beta'(s_i) \times c_{t_i} \mu(t_i) \\ \partial_t m(t_i, s_i) &= c_{t_i} (-\alpha'(t_i) - \lambda_i \mu'(t_i) + A_{t_i}(\delta(t_i, s_i) - \lambda_i \mu(t_i))) \times c_{t_i} \mu(t_i), \end{aligned}$$

and using the isometry between ν_t and \mathbb{R}^3 ,

$$D = \lambda_i \|n_{t_i} \mu(t_i)\|^2 \det(n_{t_i} \beta'(s_i), n_{t_i} (\alpha'(t_i) + \lambda_i \mu'(t_i) - A_{t_i}(\delta(t_i, s_i) - \lambda_i \mu(t_i))), n_{t_i} \mu(t_i)).$$

On the other hand, if we suppose that $\beta(s_i) = \chi(t_i, \lambda_i, x^i)$ for $x^i \in \mathbb{R}^{n-3}$, we have that $i(\beta, \chi; P_i)$ is equal to ± 1 depending on the sign of

$$E = \det(\beta'(s_i), \partial_t \chi(t_i, \lambda_i, x^i), \partial_\lambda \chi(t_i, \lambda_i, x^i), \partial_{x_4} \chi(t_i, \lambda_i, x^i), \dots, \partial_{x_n} \chi(t_i, \lambda_i, x^i)).$$

Now,

$$\partial_t \chi(t_i, \lambda_i, x^i) = \alpha'(t_i) + \lambda_i \mu'(t_i) + \sum_{j=4}^n x_j^i f_j'(t_i),$$

$$\partial_\lambda \chi(t_i, \lambda_i, x^i) = \mu(t_i),$$

$$\partial_{x_j} \chi(t_i, \lambda_i, x^i) = f_j(t_i),$$

and thus,

$$E = \det(n_{t_i} \beta'(s_i), n_{t_i} (\alpha'(t_i) + \lambda_i \mu'(t_i) + \sum_{j=4}^n x_j^i f_j'(t_i)), n_{t_i} \mu(t_i)).$$

Finally, note that

$$c_{t_i} \sum_{j=4}^n x_j^i f_j'(t_i) = -c_{t_i} A_{t_i} (\delta(t_i, s_i) - \lambda_i \mu(t_i)),$$

which implies the desired result. \square

3. THE SELF-LINKING NUMBER OF A CURVE WITH RESPECT TO A VECTOR BUNDLE

We shall define here the self-linking number of a smooth curve $\alpha : S^1 \rightarrow \mathbb{R}^n$ with respect to a vector bundle ν , as the linking number of α and $\tilde{\alpha}$, where $\tilde{\alpha} : S^1 \rightarrow \mathbb{R}^n$ is close enough to α and so that the conditions of Definition 2.3 are satisfied. To ensure that there exists such a curve $\tilde{\alpha}$, we need to assume that $\alpha(s) - \alpha(t) \notin \nu_t^\perp$, for $s \neq t$. Moreover, we also have to put some regularity conditions between the curve and the fiber bundle on the diagonal $s = t$.

Throughout this section, we will suppose that $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a smooth closed curve in \mathbb{R}^n and that ν is a smooth 3-dimensional oriented vector subbundle of the trivial vector bundle, as in Section 2.

Lemma 3.1. *Suppose that α, ν satisfy the following conditions:*

1. *For any $s \neq t$, $\alpha(s) - \alpha(t) \notin \nu_t^\perp$.*
2. *There exists $1 \leq k \leq n-2$ such that for any $t \in S^1$,*
 - (a) *$\alpha'(t), \dots, \alpha^{(k+1)}(t)$ are linearly independent;*
 - (b) *$\alpha'(t), \dots, \alpha^{(k-1)}(t) \in \nu_t^\perp$;*
 - (c) *$\langle \alpha^{(k)}(t) \rangle \oplus \langle \alpha^{(k+1)}(t) \rangle \oplus \nu_t^\perp$.*

Then, there is $\delta_0 > 0$ such that $\alpha(s) - \alpha_\delta(t) \notin \nu_t^\perp$, for any $0 < \delta < \delta_0$ and for any $(t, s) \in S^1 \times S^1$, where $\alpha_\delta(t) = \alpha(t) + \delta \alpha^{(k)}(t)$.

Proof. Suppose that this is not true. Then, for each $m \geq 1$, there are $\delta_m < 1/m$ and pairs $(t_m, s_m) \in S^1 \times S^1$ such that $\alpha(s_m) - \alpha(t_m) - \delta_m \alpha^{(k)}(t_m) \in \nu_{t_m}^\perp$. By taking subsequences if necessary, we can suppose that $s_m \rightarrow s_0 \in S^1$ and $t_m \rightarrow t_0$. If $s_0 \neq t_0$, we arrive to $\alpha(s_0) - \alpha(t_0) \in \nu_{t_0}^\perp$, in contradiction with condition 1. Otherwise, let $s_0 = t_0$. If we denote by $\{f_i(t)\}_{i=4}^n$ a frame for ν_t^\perp , we have for any $m \geq 1$,

$$(\alpha(s_m) - \alpha(t_m)) \wedge \alpha^{(k)}(t_m) \wedge f_4(t_m) \wedge \dots \wedge f_n(t_m) = 0.$$

Since

$$\alpha(s_m) = \alpha(t_m) + \sum_{j=1}^{k+1} \frac{\alpha^{(j)}(t_m)}{j!} (s_m - t_m)^j + O((s_m - t_m)^{k+2}),$$

we have after substitution and division by $(s_m - t_m)^{k+1}$,

$$\alpha^{(k+1)}(t_m) \wedge \alpha^{(k)}(t_m) \wedge f_4(t_m) \wedge \cdots \wedge f_n(t_m) + O(s_m - t_m) = 0.$$

This would imply that

$$\alpha^{(k+1)}(t_0) \wedge \alpha^{(k)}(t_0) \wedge f_4(t_0) \wedge \cdots \wedge f_n(t_0) = 0,$$

in contradiction with condition 2.(c). \square

Remark 3.2. When $n = 3$, necessarily $k = 1$ and $\nu_t^\perp = \{0\}$. Thus, conditions 1 and 2 of Lema 3.1 just say that α is embedded and that $\alpha'(t), \alpha''(t)$ are linearly independent, for any $t \in S^1$.

Definition 3.3. Suppose that α, ν satisfy conditions 1 and 2 of Lemma 3.1 and consider $\alpha_\delta(t) = \alpha(t) + \delta \alpha^{(k)}(t)$. The *self-linking number of α with respect to ν* is defined as

$$SL_\nu(\alpha) = \lim_{\delta \rightarrow 0} L_\nu(\alpha_\delta, \alpha).$$

Note that since the linking number is invariant under homotopies, Lemma 3.1 ensures that $L_\nu(\alpha_\delta, \alpha)$ does not depend on δ , if δ is small enough.

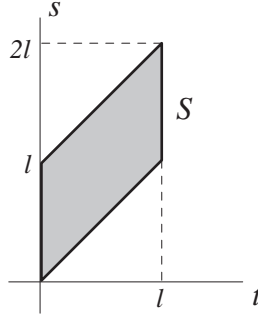


FIGURE 2.

We would like now to obtain an integral expression for the self-linking number analogous to the integral expression which defines the linking number of two curves. The first step should be to define the map e_1 . Let S be the following subset of \mathbb{R}^2 (see Figure 2) :

$$S = \{(t, s) \in \mathbb{R}^2 : 0 \leq t \leq \ell, t \leq s \leq t + \ell\}.$$

We define the map $e_1 : S \rightarrow S^2$ as follows:

$$e_1(t, s) = \begin{cases} \frac{c_t(\alpha(s) - \alpha(t))}{\|c_t(\alpha(s) - \alpha(t))\|}, & \text{if } t < s < t + \ell, \\ \frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|}, & \text{if } s = t, \\ (-1)^k \frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|}, & \text{if } s = t + \ell. \end{cases}$$

Note that e_1 is well defined when α satisfies conditions 1 and 2 of Lemma 3.1. Moreover, by taking a Taylor expansion in a neighbourhood of (t, t) or $(t, t + \ell)$, it is easy to see that e_1 is smooth.

3.1. The case k even. If k is even, the map e_1 can be considered as a map from $[0, \ell] \times S^1$ to S^2 . Moreover, $e_1^* \Omega_2$ defines a closed 2-form on $S^1 \times S^1$, which is the limit when $\delta \rightarrow 0$ of the closed 2-form associated to the pair (α_δ, α) in the definition of the linking number. This gives the following result.

Proposition 3.4. *Suppose that α, ν satisfy conditions 1 and 2 of Lemma 3.1 for k even. Then,*

$$SL_\nu(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2.$$

3.2. The case k odd. This case is more complicated. Let S^0 denote the open subset of S given by the pairs (t, s) such that $t < s < t + \ell$ and $c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t) \neq 0$. Then we can complete e_1 on S^0 in order to get a frame of \mathbb{R}^3 as we did in the proof of Proposition 2.7. If $t < s < t + \ell$, we define

$$e_3(t, s) = \frac{c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t)}{\|c_t(\alpha(s) - \alpha(t)) \times c_t \alpha^{(k)}(t)\|}.$$

Moreover, it is possible to extend e_3 smoothly to the boundaries $s = t$ and $s = t + \ell$. In fact, by taking a Taylor expansion in a neighbourhood of (t, t) or $(t, t + \ell)$, we get that

$$e_3(t, t) = e_3(t, t + \ell) = \frac{c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)\|}.$$

We define e_2 in the obvious way, $e_2(t, s) = e_3(t, s) \times e_1(t, s)$. Finally, we define the 1-forms $\omega_{ij} = \langle de_i, e_j \rangle$, for any $i, j = 1, 2, 3$.

Proposition 3.5. *Suppose that α, ν satisfy conditions 1 and 2 of Lemma 3.1 for k odd. Then,*

$$SL_\nu(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 - \frac{1}{2\pi} \int_{S^1} \phi,$$

where $\phi(t) = \omega_{32}(t, t)$.

Proof. Let $\{f_i(t)\}_{i=4}^n$ be an orthonormal oriented frame of ν_t^\perp and consider the map $\chi : S^1 \times \mathbb{R} \times \mathbb{R}^{n-3} \rightarrow \mathbb{R}^n$ given by

$$\chi(t, \lambda, x_4, \dots, x_n) = \alpha(t) + \lambda \alpha^{(k)}(t) + \sum_{j=4}^n x_j f_j(t).$$

By the Transversality Theorem, we have that for a residual subset of curves α and vector bundles ν with the corresponding C^∞ Whitney topologies, the curve α meets the hypersurface χ transversely at a finite number of points:

$$P_i = \alpha(s_i) = \alpha(t_i) + \lambda_i \alpha^{(k)}(t_i) + \sum_{j=4}^n x_j^i f_j(t_i), \quad i = 1, \dots, N,$$

with $s_i \neq t_i$ and $\lambda_i \neq 0$. Since $SL_\nu(\alpha)$, $\int_{S^1 \times S^1} e_1^* \Omega_2$ and $\int_{S^1} \phi$ depend continuously on α and ν , we can suppose that α and ν are generic in the above sense.

In particular, for δ small enough, the same can be said if we consider the intersection of α with χ_δ , where

$$\chi_\delta(t, \lambda, x_4, \dots, x_n) = \alpha_\delta(t) + \lambda \alpha^{(k)}(t) + \sum_{j=4}^n x_j f_j(t).$$

Then, Proposition 2.7 gives that

$$L_\nu(\alpha_\delta, \alpha) = \frac{1}{2} \sum_{i=1}^N \operatorname{sgn}(\lambda_i + \delta) i(\alpha, \chi_\delta; P_i),$$

and taking limit when $\delta \rightarrow 0$,

$$SL_\nu(\alpha) = \frac{1}{2} \sum_{i=1}^N \operatorname{sgn}(\lambda_i) i(\alpha, \chi; P_i).$$

On the other hand, note that $S^0 = S \setminus \{(t_1, s_1), \dots, (t_N, s_N)\}$. By using the same argument as in the proof of Proposition 2.7, $d\omega_{32} = \omega_{12} \wedge \omega_{13} = e_1^* \Omega_2$. If we apply Stokes Theorem,

$$\frac{1}{4\pi} \int_S e_1^* \Omega_2 = \frac{1}{4\pi} \int_{\partial S} \omega_{32} + \frac{1}{4\pi} \lim_{\epsilon \rightarrow 0} \sum_{i=1}^N \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32},$$

where $D_\epsilon(t_i, s_i)$ denotes the disk centered at (t_i, s_i) of radius $\epsilon > 0$ in S . Again we refer to the proof of Proposition 2.7 to claim that

$$\frac{1}{2\pi} \lim_{\epsilon \rightarrow 0} \int_{\partial D_\epsilon(t_i, s_i)} \omega_{32} = \operatorname{sgn}(\lambda_i) i(\alpha, \chi; P_i).$$

In particular,

$$SL_\nu(\alpha) = \frac{1}{4\pi} \int_S e_1^* \Omega_2 - \frac{1}{4\pi} \int_{\partial S} \omega_{32}.$$

To conclude the proof, we just have to compute the integral on ∂S . We parameterize ∂S by considering the curves: $\gamma_1(u) = (0, u)$, $\gamma_2(u) = (u, u + \ell)$, $\gamma_3(u) = (\ell, u + \ell)$ and $\gamma_4(u) = (u, u)$, for $u \in [0, \ell]$. Then, we have that

$$\int_{\partial S} \omega_{32} = \int_0^\ell \omega_{32}(-\gamma_1' - \gamma_2' + \gamma_3' + \gamma_4') du.$$

Note that $\omega_{32}(0, u) = \omega_{32}(\ell, u + \ell)$ and $\omega_{32}(u, u) = -\omega_{32}(u, u + \ell)$ for any $u \in [0, \ell]$. This gives that

$$\int_{\partial S} \omega_{32} = 2 \int_{S^1} \phi.$$

□

4. THE ORTHOGONAL SELF-LINKING NUMBER

We consider here the case that the vector bundle ν is equal to the orthogonal vector bundle of the curve. That is, ν_t is the 3-plane orthogonal to the subspace generated by the $n - 3$ first derivatives of the curve.

Definition 4.1. Let $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a closed smooth curve in \mathbb{R}^n and suppose that:

1. For any $t \in S^1$, $\alpha'(t), \alpha''(t), \dots, \alpha^{(n-1)}(t)$ are linearly independent. In this way, at each point there is a well defined Frenet frame $\{f_i(t)\}_{i=1}^n$ and also we have the curvatures $\{\kappa_i(t)\}_{i=1}^{n-1}$. The *orthogonal vector bundle* ν is defined so that ν_t is the 3-plane generated by $f_{n-2}(t), f_{n-1}(t), f_n(t)$.
2. For any $s \neq t$ in S^1 , $\alpha(s) - \alpha(t) \notin \nu_t^\perp$. That is, the $(n-3)$ -osculating plane at t does not meet the curve at any other point.

It follows that α, ν satisfy conditions 1 and 2 of Lemma 3.1 for $k = n-2$. The self-linking number of α with respect to the orthogonal vector bundle will be called the *orthogonal self-linking number* and will be denoted by $SL^\perp(\alpha)$.

With respect to this orthogonal vector bundle, we have that the orthogonal projection $n_t : \mathbb{R}^n \rightarrow \nu_t$ is given by

$$n_t(x) = \langle x, f_{n-2}(t) \rangle f_{n-2}(t) + \langle x, f_{n-1}(t) \rangle f_{n-1}(t) + \langle x, f_n(t) \rangle f_n(t).$$

We also need to know about the linear map $A_t : \mathbb{R}^n \rightarrow \nu_t$. To simplify computations, we will suppose that α is parameterized by arc length. Then,

$$A_t(x) = \sum_{i=n-2}^n \langle x, \alpha'_i(t) \rangle f_i(t) = -\langle x, f_{n-3}(t) \rangle \kappa_{n-3}(t) f_{n-2}(t).$$

With this we can easily compute the 2-form $e_1^* \Omega_2$ used in the integral formula of the self-linking number. But when $k = n-2$ is odd, we also need to compute the 1-form ϕ on S^1 given by $\phi(t) = \omega_{32}(t, t)$. Note that

$$\begin{aligned} e_1(t, t) &= \frac{c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k)}(t)\|} = c_t f_{n-2}(t), \\ e_3(t, t) &= \frac{c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)}{\|c_t \alpha^{(k+1)}(t) \times c_t \alpha^{(k)}(t)\|} = -c_t f_n(t), \\ e_2(t, t) &= e_3(t, t) \times e_1(t, t) = -c_t f_{n-1}(t), \\ \omega_{32}(t, t) &= \langle de_3(t, t), e_2(t, t) \rangle = -\kappa_{n-1}(t) dt. \end{aligned}$$

Thus, we have the following integral expression for the orthogonal self-linking number.

Corollary 4.2. *Let $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a closed smooth curve in \mathbb{R}^n satisfying conditions 1 and 2 of Definition 4.1. Then*

$$e_1^* \Omega_2 = \frac{\langle \delta(t, s), f_{n-3}(t) \rangle \kappa_{n-3}(t) \det(n_t \delta(t, s), n_t \alpha'(s), f_{n-2}(t))}{\|n_t \delta(t, s)\|^3} dt \wedge ds,$$

where $\delta(t, s) = \alpha(s) - \alpha(t)$. Moreover, the orthogonal self-linking number of α is equal to

$$SL^\perp(\alpha) = \begin{cases} \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2, & \text{when } n \text{ is even,} \\ \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 + \frac{1}{2\pi} \int_{S^1} \kappa_{n-1}(t) dt, & \text{when } n \text{ is odd.} \end{cases}$$

Given $\alpha : S^1 \rightarrow \mathbb{R}^n$ a closed smooth curve in \mathbb{R}^n satisfying conditions 1 and 2 of Definition 4.1, we can consider the osculating developable hypersurface, which is the map

$\chi^\top : S^1 \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^n$ defined by

$$\chi^\top(t, x_1, \dots, x_{n-2}) = \alpha(t) + \sum_{i=1}^{n-2} x_i f_i(t).$$

By condition 1, this is an immersion at those points such that $x_{n-2} \neq 0$. Moreover, condition 2 implies that if the curve meets this map at a point $P = \alpha(s) = \chi^\top(t, x_1, \dots, x_{n-2})$ with $s \neq t$, then necessarily $x_{n-2} \neq 0$. Note that the case $s = t$ would imply that $x_1 = \dots = x_{n-2} = 0$.

Corollary 4.3. *Suppose that α meets the map χ^\top transversely at a finite number of non-diagonal points and let $(t_1, s_1), \dots, (t_N, s_N)$ be the pairs in $S^1 \times S^1$ corresponding to these points. Then, the orthogonal self-linking number of α is equal to*

$$SL^\perp(\alpha) = -\frac{1}{2} \sum_{i=1}^N \operatorname{sgn} \langle \alpha'(s_i), f_n(t_i) \rangle.$$

5. THE OSCULATING SELF-LINKING NUMBER

Here, we look at the self-linking number of a curve with respect to its osculating vector bundle. That is, ν_t is the 3-plane generated by the 3 first derivatives of the curve.

Definition 5.1. Let $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a closed smooth curve in \mathbb{R}^n and suppose that:

1. For any $t \in S^1$, $\alpha'(t), \alpha''(t), \alpha'''(t)$ are linearly independent. We will denote by ν the osculating vector bundle of α , that is, $\nu_t = \langle \alpha'(t), \alpha''(t), \alpha'''(t) \rangle$.
2. For any $s \neq t$ in S^1 , $\alpha(s) - \alpha(t) \notin \nu_t^\perp$. That is, the $(n-3)$ -orthogonal plane at t does not meet the curve at any other point.

In this case, α, ν satisfy conditions 1 and 2 of Lemma 3.1 for $k = 1$. Thus, we define the *osculating self-linking number*, $SL^\top(\alpha)$, as the self-linking number of α with respect to the osculating vector bundle.

Now, we use $f_1(t), f_2(t), f_3(t)$ for the (partial) Frenet frame of ν_t and $\kappa_1(t), \kappa_2(t)$ for the non-vanishing curvatures. Then, $n_t : \mathbb{R}^n \rightarrow \nu_t$ is given by

$$n_t(x) = \langle x, f_1(t) \rangle f_1(t) + \langle x, f_2(t) \rangle f_2(t) + \langle x, f_3(t) \rangle f_3(t).$$

Again, we will suppose for simplicity that α is parameterized by arc length. Thus,

$$A_t(x) = \sum_{i=1}^3 \langle x, o_t f'_i(t) \rangle f_i(t) = k(t) \langle x, o_t \alpha^{(4)}(t) \rangle f_3(t),$$

where $k(t) = 1/\kappa_1(t)\kappa_2(t)$.

Finally, we compute the 1-form $\phi(t) = \omega_{32}(t, t)$:

$$\begin{aligned} e_1(t, t) &= \frac{c_t \alpha'(t)}{\|c_t \alpha'(t)\|} = c_t f_1(t), \\ e_3(t, t) &= \frac{c_t \alpha''(t) \times c_t \alpha'(t)}{\|c_t \alpha''(t) \times c_t \alpha'(t)\|} = -c_t f_3(t), \\ e_2(t, t) &= e_3(t, t) \times e_1(t, t) = -c_t f_2(t), \\ \omega_{32}(t, t) &= \langle de_3(t, t), e_2(t, t) \rangle = -\kappa_2(t) dt. \end{aligned}$$

Thus, we have the following integral expression for the osculating self-linking number.

Corollary 5.2. *Let $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a closed smooth curve in \mathbb{R}^n satisfying conditions 1 and 2 of Definition 5.1. Then*

$$e_1^* \Omega_2 = \frac{\det(n_t \delta(t, s), n_t \alpha'(s), \alpha'(t) - k(t) \langle \delta(t, s), o_t \alpha^{(4)}(t) \rangle f_3(t))}{\|n_t \delta(t, s)\|^3} dt \wedge ds,$$

where $\delta(t, s) = \alpha(s) - \alpha(t)$. Moreover, the osculating self-linking number of α is equal to

$$SL^\top(\alpha) = \frac{1}{4\pi} \int_{S^1 \times S^1} e_1^* \Omega_2 + \frac{1}{2\pi} \int_{S^1} \kappa_2(t) dt.$$

Finally, we can compute the osculating self-linking number by looking at the intersection of the curve with its orthogonal developable. Let $\{f_j(t)\}_{j=4}^n$ be any orthonormal oriented frame that trivializes ν_t^\perp . We consider the orthogonal developable hypersurface, which is the map $\chi^\perp : S^1 \times \mathbb{R}^{n-2} \rightarrow \mathbb{R}^n$ defined by

$$\chi^\perp(t, x_3, \dots, x_n) = \alpha(t) + \sum_{i=3}^n x_i f_i(t).$$

Since in this case $k = 1$, we have by Definition 3.3 that $SL^\top(\alpha) = \lim_{\delta \rightarrow 0} L_\nu(\alpha + \delta f_1, \alpha)$. But it is not difficult to see that we obtain the same number if we change f_1 by f_2 or f_3 . Thus, we have the following immediate consequence of Proposition 2.7, for $\mu(t) = f_3(t)$.

Corollary 5.3. *Let $\alpha : S^1 \rightarrow \mathbb{R}^n$ be a closed smooth curve in \mathbb{R}^n satisfying conditions 1 and 2 of Definition 5.1. Suppose that α meets the map χ^\perp transversely at a finite number of non-diagonal points and let*

$$P_i = \alpha(s_i) = \alpha(t_i) + \sum_{j=3}^n x_j^i f_j(t_i), \quad i = 1, \dots, N$$

be those points. Then, the osculating self-linking number of α is equal to

$$SL^\top(\alpha) = \frac{1}{2} \sum_{i=1}^N \operatorname{sgn}(x_3^i) i(\alpha, \chi^\perp; P_i).$$

6. THE EXAMPLES

In this last section, we will give some examples which show that when $n > 3$, the orthogonal and the osculating self-linking numbers are not trivial and are independent. All the examples are in \mathbb{R}^4 and the computations have been done with *Mathematica* [3]. We compute the intersection of the curve with χ^\top or χ^\perp and the corresponding indices. Moreover, we also compute the integral value of SL^\perp or SL^\top in order to ratify the results.

Example 6.1. Let $\alpha : S^1 \rightarrow \mathbb{R}^4$ be the curve given by

$$\alpha(t) = \left(\cos(A + t) + \sin^2(t), \cos(A + 2t), \cos(t), \frac{A \sin(3t)}{27} \right).$$

It follows that for $A = 1$ and $A = 1.3$, the curve α satisfies conditions 1 and 2 of Definition 4.1 and Definition 5.1.

When $A = 1$, α meets χ^\perp transversely at four points with indices $1, 1, 1, -1$. In fact, we compute numerically the integral of Corollary 4.2 and obtain that $SL^\top(\alpha) = 1$. If we look now at the intersection with χ^\top , there are just two points of transverse intersection, both with index 1. In this case, the integral formula of Corollary 5.2 gives $SL^\perp(\alpha) = 1$.

When $A = 1.3$, the intersection with χ^\perp gives again four points with indices 1, 1, 1, -1 and the numerical value of the integral formula is $SL^\top(\alpha) = 1$. However, although there are two points of transverse intersection with χ^\top , this time the indices are 1, -1 and the integral formula gives in this case $SL^\perp(\alpha) = 0$.

Example 6.2. We consider now a different family of curves in \mathbb{R}^4 :

$$\alpha(t) = \left(-\cos(A+t) + \frac{A \sin(2t)}{8}, \frac{-A^3 \cos(2t)}{8} + \sin(A+t), \frac{\sin(5t)}{125}, \frac{A^2 \sin(3t)}{27} \right).$$

For $A = 1.6$, α satisfies conditions 1 and 2 of Definition 4.1 and Definition 5.1. The intersection with χ^\perp is equal to six points, all of them having index 1, and the numerical computation of the integral formula gives $SL^\top(\alpha) = 3$. The intersection with χ^\top is also equal to six points, but in this case two of them have index 1 and the other four -1 . The integral formula gives $SL^\perp(\alpha) = -1$.

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